• Recall when changing variables from cartesian into polar, cylindrical or spherical coordinates the differentials satisfy

Polar: \( dA = \) 

Cylindrical: \( dV = \) 

Spherical: \( dV = \)

These are special cases of a broad family of changes of variables.

• If \( G(u, v) = (x(u, v), y(u, v)) \) is a mapping then the **Jacobian** of \( G \) is the determinant

\[
\text{Jac}(G) = \frac{\partial (x, y)}{\partial (u, v)} =
\]

• If \( F = G^{-1} \) is the inverse of \( G \) then \( \text{Jac}(F) = \).

• **Change of Variables Formula:** If \( F : D_0 \to D \) has component functions with continuous partial derivatives and one-to-one on the interior of \( D_0 \), and if \( f \) is continuous, then

\[
\int \int_D f(x, y) \, dx \, dy = dudv
\]

• This generalizes to three variables, if \( G(u, v, w) = (x(u, v, w), y(u, v, w), x(u, v, w)) \) is a mapping (satisfying some technical conditions) then

\[
\text{Jac}(G) = \frac{\partial (x, y, z)}{\partial (u, v, w)} = \text{ and } dx \, dy \, dz =
\]
1. Verify the change of variables for polar, cylindrical and spherical coordinates.

2. A map is linear if it has the form \( G(u, v) = (Au + Bv, Cu + Dv) \) where \( A, B, C, D \) are constants. Compute the Jacobian for such a \( G \) and verify that it is constant.

3. Use the transformation \( \Phi(u, v) = \left( \frac{u}{v + 1}, \frac{uv}{v + 1} \right) \) to compute \( \iint_{D} (x + y) \, dx \, dy \), where \( D \) is the region bounded by \( y = 2x, \ y = x, \ y = 3 - x, \) and \( y = 6 - x \).

4. Find a mapping \( \Phi \) that maps the unit disk \( x^2 + y^2 \leq 1 \) onto the ellipse \( \left( \frac{x}{a} \right)^2 + \left( \frac{y}{b} \right)^2 \leq 1 \). Then use the change of variables formula to prove that the area of the ellipse is \( \pi ab \).

5. Compute the area of the region \( x^2 + 2xy + 2y^2 - 4y \leq 8 \) as an integral in the variables \( u = x + y \) and \( v = y \).
Recall when changing variables from cartesian into polar, cylindrical or spherical coordinates the differentials satisfy

- **Polar:** \( dA = r \, dr \, d\theta \)
- **Cylindrical:** \( dV = r \, dr \, d\theta \, dz \)
- **Spherical:** \( dV = \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \)

These are special cases of a broad family of changes of variables.

- If \( G(u,v) = (x(u,v), y(u,v)) \) is a mapping then the **Jacobian** of \( G \) is the determinant

\[
\text{Jac}(G) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}
\]

- If \( F = G^{-1} \) is the inverse of \( G \) then \( \text{Jac}(F) = \frac{1}{\text{Jac}(G)} \).

- **Change of Variables Formula:** If \( F : D_0 \to D \) has component functions with continuous partial derivatives and one-to-one on the interior of \( D_0 \), and if \( f \) is continuous, then

\[
\iint_D f(x,y) \, dxdy = \iint_{D_0} f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \, du \, dv
\]

- This generalizes to three variables, if \( G(u,v,w) = (x(u,v,w), y(u,v,w), z(u,v,w)) \) is a mapping (satisfying some technical conditions) then

\[
\text{Jac}(G) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} \quad \text{and} \quad dx \, dy \, dz = \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| \, du \, dv \, dw
1. Verify the change of variables for polar, cylindrical and spherical coordinates.

2. A map is **linear** if it has the form \( G(u, v) = (Au + Bv, Cu + Dv) \) where \( A, B, C, D \) are constants. Compute the Jacobian for such a \( G \) and verify that it is constant.

3. Use the transformation \( \Phi(u, v) = \left( \frac{u}{v + 1}, \frac{uv}{v + 1} \right) \) to compute \( \iint_D (x + y) \, dx \, dy \), where \( D \) is the region bounded by \( y = 2x, y = x, y = 3 - x, \) and \( y = 6 - x \).

**Answer:** \( \frac{21}{2} \).

4. Find a mapping \( \Phi \) that maps the unit disk \( x^2 + y^2 \leq 1 \) onto the ellipse \( \left( \frac{x}{a} \right)^2 + \left( \frac{y}{b} \right)^2 \leq 1 \). Then use the change of variables formula to prove that the area of the ellipse is \( \pi ab \).

**Answer:** Use \( \Phi(u, v) = (au, bv) \).

5. Compute the area of the region \( x^2 + 2xy + 2y^2 - 4y \leq 8 \) as an integral in the variables \( u = x + y \) and \( v = y \).

**Answer:** \( 12\pi \).